On commutator-inversion invariant conditions and related gyrogroup constructions

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19th International Conference of Young Algebraists in Thailand
Mahasarakham University, Mahasarakham, Thailand

March 25–27, 2026

Outline

Topics

In today's talk, we will cover the following topics:

1. Introduction to nilpotent groups and 2-Engel groups
2. Introduction to gyrogroups
3. Constructions of a gyrogroup from a group
4. CII and CCII groups
5. Concrete examples

Commutators and abelian groups

Let G be a group. For each pair (g, h) of elements in G, the *commutator* of g and h is defined as

$$[g, h] = g^{-1}h^{-1}gh. \tag{1}$$

Note that G is abelian if and only if $[g, h] = e$ for all g and h in G. That is, G is abelian if and only if the *derived subgroup*

$$G' = \langle [g, h] : g, h \in G \rangle \tag{2}$$

is trivial. This is a characterization of abelian groups.

Definition of nilpotent groups

Nilpotent groups play an important role in group theory and arise naturally in many areas of mathematics. Informally, a nilpotent group is a group whose commutators eventually become trivial when iterated sufficiently many times.

More precisely, a group G is said to be *nilpotent* if its lower central series $G = \gamma_1(G) \geq \gamma_2(G) \geq \gamma_3(G) \geq \cdots$, defined recursively by

$$\gamma_{i+1}(G) = [\gamma_i(G), G] = \langle [g, h] : g \in \gamma_i(G), h \in G \rangle,$$

terminates at the trivial subgroup after finitely many steps. The smallest integer c such that $\gamma_{c+1}(G) = \{e\}$ is called the *nilpotency class* of G.

Examples of nilpotent groups

Nilpotent groups may be viewed as a natural generalization of abelian groups since every abelian group is nilpotent of class 1. Nilpotent groups also appear in the study of Lie groups, algebraic groups, and various areas of algebra. Because of their rich structure and useful properties, nilpotent groups play a central role in both finite and infinite group theory.

Prime examples of nilpotent groups include

- abelian groups;
- finite p-groups;
- Heisenberg groups.

Heisenberg groups

The *continuous Heisenberg group* is the group of matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

where x, y, and z are real numbers, under usual matrix multiplication.

The continuous Heisenberg group is nilpotent of class 2. If we choose the parameters x, y, and z from $\mathbb{Z}$, we obtain the *discrete Heisenberg group*. More generally, the parameters x, y, and z can be chosen from a ring with identity.

Definition of 2-Engel groups

A 2-Engel group is a group in which the 2-Engel identity holds for all pairs of elements. More precisely, a group G is called a *2-Engel group* if

$$[[g, h], h] = e \tag{3}$$

for all $g, h \in G$. The 2-Engel identity (3) provides a natural setting for studying how commutator identities control group structure.

2-Engel groups play an important role in the broader theory of Engel conditions, which are central in the investigation of nilpotency and solvability in group theory.

Levi's result

It turns out that the family of 2-Engel groups is included in the family of nilpotent groups by Levi's result [1].

Theorem 1 (Levi, 1942)

Every 2-Engel group is nilpotent of class at most 3.

[1] F. Levi, Groups in which the commutator operation satisfies certain algebraic conditions, *J. Indian Math. Soc.*, **6** (1942) 87–97.

Characterizations of 2-Engel groups

Let G be a group. Then the following statements are equivalent:

1. G is a 2-Engel group.
2. x commutes with gxg^{-1} for all $g, x \in G$.
3. x commutes with $[g, x]$ for all $g, x \in G$.
4. The normal closure of any cyclic subgroup in G is abelian.
5. For all $g, x \in G$, the subgroup $\langle g, x \rangle$ is a nilpotent group of class at most 2.

Historical remarks

A gyrogroup is a non-associative algebraic structure introduced by Ungar [2] to model the algebra underlying relativistic velocity addition in special relativity. It generalizes the notion of a group by replacing the usual associativity law with a weaker condition called the gyroassociative law, which is controlled by special automorphisms known as gyrations.

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[2] A. Ungar, Gyrogroups, Sixth International Conference on Geometry, *J. Geom.* **44** (1992), 21–22.

Equivalent notion of gyrogroups

It turns out that the family of gyrogroups coincides with the family of left Bol loops with the A_ℓ-property. In fact, a *left Bol loop* is a loop L satisfying the *left Bol identity*

$$x(y(xz)) = (x(yx))z \tag{4}$$

for all $x, y, z \in L$. A left Bol loop is said to have the *A_ℓ-property* if every left inner mapping of L is an automorphism of the loop.

Definition of gyrogroups

Let G be a non-empty set, and let $\oplus$ be a binary operation on G. Then $(G, \oplus)$ is called a *gyrogroup* if the following statements hold:

1. $\exists e \in G\ \forall a \in G, a \oplus e = a = e \oplus a$ (identity element)
2. $\forall a \in G\, \exists b \in G, b \oplus a = e = a \oplus b$ (inverse element)
3. $\forall a, b \in G\ \exists \mathrm{gyr}[a, b], \mathrm{gyr}[b, a] \in \mathrm{Aut}(G, \oplus)$ such that
   - $a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a, b](c)$ (left gyroassociative law)
   - $(a \oplus b) \oplus c = a \oplus (b \oplus \mathrm{gyr}[b, a](c))$ (right gyroassociative law)
4. $\forall a, b \in G$,
   - $\mathrm{gyr}[a \oplus b, b] = \mathrm{gyr}[a, b]$ (left loop property)
   - $\mathrm{gyr}[a, b \oplus a] = \mathrm{gyr}[a, b]$ (right loop property)

Concrete example of a gyrogroup

Let $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$. *Möbius addition* [3], denoted by $\oplus_M$, is defined as

$$a \oplus_M b = \frac{a+b}{1+\bar{a}b} \tag{5}$$

for all $a, b \in \mathbb{D}$. Then $(\mathbb{D}, \oplus_M)$ forms a gyrogroup, called the (complex) *Möbius gyrogroup*. In this case, the gyroautomorphism generated by a and b is the rotation map given by

$$\mathrm{gyr}[a,b](z) = \frac{1+a\bar{b}}{1+\bar{a}b}z \tag{6}$$

for all $z \in \mathbb{D}$.

The Möbius gyrogroup is one of the most important examples of gyrogroups and plays a fundamental role in the development of gyrogroup theory.

[3] A. Ungar, The holomorphic automorphism group of the complex disk, *Aequationes Math.* **47** (1994), 240–254.

Relationships between groups and gyrogroups

Recall the gyroassociative law:

$$\begin{aligned} a \oplus (b \oplus c) &= (a \oplus b) \oplus \mathrm{gyr}[a, b](c); \\ (a \oplus b) \oplus c &= a \oplus (b \oplus \mathrm{gyr}[b, a](c)). \end{aligned}$$

- Every group can be viewed as a gyrogroup by defining the gyroautomorphisms to be the identity automorphism.
- Any gyrogroup with trivial gyroautomorphisms forms a group.

Gyrotriples

A subset B of a group Γ is a *twisted subgroup* of Γ if (i) $e \in B$, e being the identity of Γ; (ii) $b \in B$ implies $b^{-1} \in B$; and (iii) $a, b \in B$ implies $aba \in B$.

A subset B of a group Γ is a (left) *transversal* to a subgroup Ξ of Γ if each element g of Γ can be written uniquely as $g = bh$ for some $b \in B, h \in \Xi$.

Definition 2

Let Γ be a group, let B be a subset of Γ, and let Ξ be a subgroup of Γ. A triple (Γ, B, Ξ) is called a *gyrotriple* if the following statements hold:

1. B is a transversal to Ξ in Γ;
2. B is a twisted subgroup of Γ;
3. Ξ normalizes B, that is, $hBh^{-1} \subseteq B$ for all $h \in \Xi$.

Gyrogroup → Group

Let G be a gyrogroup. For each $a \in G$, the *left gyrotranslation* L_a is a map defined by $L_a(x) = a \oplus x$ for all $x \in G$. The set of all left gyrotranslations is denoted by $\hat{G}$. It is known that $\Sigma = \{L_a \circ \alpha \colon a \in G, \alpha \in \mathrm{Aut}(G)\}$ forms a group under composition of maps with group law:

$$(L_a \circ \alpha) \circ (L_b \circ \beta) = L_{a \oplus \alpha(b)} \circ (\mathrm{gyr}[a, \alpha(b)] \circ \alpha \circ \beta) \tag{7}$$

for all $a, b \in G, \alpha, \beta \in \mathrm{Aut}(G)$. Furthermore, $\hat{G} \subseteq \Sigma$ and $\mathrm{Aut}(G)$ is a subgroup of Σ.

Theorem 3

If G is a gyrogroup, then $(\Sigma, \hat{G}, \mathrm{Aut}(G))$ is a gyrotriple.

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[4] T. Suksumran, Involutive groups, unique 2-divisibility, and related gyrogroup structures, *J. Algebra Appl.*, **16**, No. 6 (2017), Article 1750114 (22 pages).

Group $\to$ Gyrogroup

Suppose that a subset B of a group Γ is a transversal to a subgroup Ξ of a group Γ. By definition, for all $a, b \in B$, there are unique elements $a \odot b \in B$ and $h(a, b) \in \Xi$ such that $ab = (a \odot b)h(a, b)$. In some cases, the transversal operation $\odot$ becomes a gyrogroup operation.

Theorem 4 (Foguel & Ungar, 2000 • S., 2017)

Let (Γ, B, Ξ) be a gyrotriple. Then B is a gyrogroup under the transversal operation. In this case, the group identity of Γ acts as the gyrogroup identity of B, and $\ominus b = b^{-1}$ for all $b \in B$. The gyroautomorphism of B generated by a and b is the restriction of conjugation by $h(a, b)$ for all $a, b \in B$.

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[5] T. Foguel and A. Ungar, Involutory decomposition of groups into twisted subgroups and subgroups, *J. Group Theory* **3** (2000), 27–46.

Operation $\oplus_\kappa$ defined on a group

Let Γ be a group. Define a binary operation $\oplus_\kappa$ on the same underlying set of Γ by

$$a \oplus_\kappa b = a^2ba^{-1} \tag{8}$$

for all $a, b \in \Gamma$. The problem of determining when $\oplus_\kappa$ defines a gyrogroup operation leads to a new characterization of 2-Engel groups.

Commutator-Inversion invariant condition

In any group, the *commutator-inversion invariant condition* is defined as

$$[g, h] = [g^{-1}, h^{-1}]. \tag{9}$$

A group Γ is called *commutator-inversion invariant* (or simply a *CII group*) if Condition (9) is satisfied for all $g, h \in \Gamma$.

Theorem 5 (Characterization of 2-Engel groups)

A group Γ is 2-Engel if and only if $[g, h] = [g^{-1}, h^{-1}]$ for all $g, h \in \Gamma$.

[6] T. Suksumran, On commutator-inversion invariant groups and gyrogroup construction, *J. Algebra Appl.*, **23**, No. 3 (2024), Article 2450042 (21 pages).

Being central by a CII group

To obtain a gyrotriple from a generic group, we define a new family of groups.

Definition 6

A group Γ is *central by a commutator-inversion invariant group* (or simply a *CCII group*) if the central quotient $\Gamma/Z(\Gamma)$ is a commutator-inversion invariant group. Here, $Z(\Gamma)$ denotes the center of Γ.

CCII group $\to$ gyrogroup

The condition of being a CCII group is used to construct a gyrogroup under the operation $\oplus_\kappa$.

Theorem 7 (S., 2024)

If Γ is a CCII group, then the underlying set of Γ can be made into a gyrogroup, denoted by Γ^{gyr}, under the operation

$$a \oplus_\kappa b = a^2ba^{-1}$$

for all $a, b \in \Gamma$. In this case, the gyrogroup identity of Γ^{gyr} coincides with the group identity of Γ, and $\ominus a = a^{-1}$ for all $a \in \Gamma$. Furthermore, the gyroautomorphism of Γ^{gyr} generated by a and b is conjugation by the commutator $[a^{-1}, b]$.

Gyrogroup $\to$ CCII group

The converse of Theorem 7 also holds in the sense of the following theorem.

Theorem 8 (S., 2024)

Let Γ be a group. Define $\oplus_\kappa$ as in Equation (8), and define a map γ by sending (a, b) to conjugation by $[a^{-1}, b]$ for all $a, b \in \Gamma$. If $(\Gamma, \oplus_\kappa)$ forms a gyrogroup with γ as the gyrator map, then Γ is a CCII group.

Degenerate case

We identify a condition under which the gyrogroup constructed via Theorem 7 has trivial gyroautomorphisms and hence reduces to a group.

Theorem 9

Let Γ be a CCII group. Then, every gyroautomorphism of Γ^{gyr} is trivial if and only if Γ is nilpotent of class at most 2.

Properties of CII groups

Some algebraic properties of CII groups are listed below.

Theorem 10

Let Γ be a group.

1. *If Γ is CII, then Γ is CCII.*
2. *If Γ is a nilpotent group of class at most 2, then Γ is CII.*
3. *Every group of exponent 3 is CII.*
4. *If Γ is finite and $3 \nmid |\Gamma|$, then Γ is CII if and only if Γ is nilpotent of class at most 2.*
5. *Every CII group is nilpotent of class at most 3.*
6. *Every finite group of order p^k, where p is a prime and $k \leq 3$, is CII.*

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[7] T. Suksumran, A note on CII groups and CCII groups, *J. Algebra Comb. Discrete Appl.*, **12**, No. 3 (2025), 259–274.

Properties of CCII groups

Some algebraic properties of CCII groups are listed below.

Theorem 11

1. *Every nilpotent group of class at most* 3 *is CCII.*
2. *Every CCII group is nilpotent of class at most* 4.
3. *Every finite group of order* p^4*, where* p *is a prime, is CCII.*

[7] T. Suksumran, A note on CII groups and CCII groups, *J. Algebra Comb. Discrete Appl.*, **12**, No. 3 (2025), 259–274.

Relationships between nilpotent, CII, and CCII groups

We may use the following diagram to verify whether a group is CII or CCII.

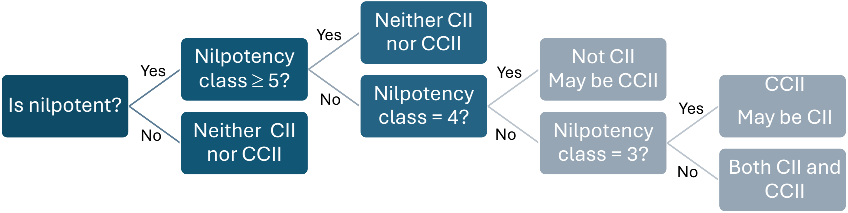

Figure 1: Diagram for verifying whether a group is CII or CCII.

Groups of order n with $n \leq 8$

We list groups of order n with $n \leq 8$ and determine whether each group is CII or CCII.

Order	Group	CII group?	CCII group?	Is nilpotent of class at most 2?
1	$\{e\}$	Yes	Yes	Yes
2	$\mathbb{Z}_2$	Yes	Yes	Yes
3	$\mathbb{Z}_3$	Yes	Yes	Yes
4	$\Lambda_4[2]$	Yes	Yes	Yes
5	$\mathbb{Z}_5$	Yes	Yes	Yes
6	$\mathbb{Z}_6$	Yes	Yes	Yes
6	S_3	No	No	No
7	$\mathbb{Z}_7$	Yes	Yes	Yes
8	$\Lambda_8[3]$	Yes	Yes	Yes
8	D_8	Yes	Yes	Yes
8	Q_8	Yes	Yes	Yes

Table 1: The groups of order n with $n \leq 8$ (up to isomorphism).

Groups of order n with $9 \leq n \leq 14$

We list groups of order n with $9 \leq n \leq 14$ and determine whether each group is CII or CCII.

Order	Group	CII group?	CCII group?	Is nilpotent of class at most 2?
9	$\Lambda_9[2]$	Yes	Yes	Yes
10	$\mathbb{Z}_{10}$	Yes	Yes	Yes
10	D_{10}	No	No	No
11	$\mathbb{Z}_{11}$	Yes	Yes	Yes
12	$\Lambda_{12}[2]$	Yes	Yes	Yes
12	A_4	No	No	No
12	D_{12}	No	No	No
12	Q_{12}	No	No	No
13	$\mathbb{Z}_{13}$	Yes	Yes	Yes
14	$\mathbb{Z}_{14}$	Yes	Yes	Yes
14	D_{14}	No	No	No

Table 2: The groups of order n with $9 \leq n \leq 14$ (up to isomorphism).

Groups of order n with $15 \leq n \leq 16$

We list groups of order n with $15 \leq n \leq 16$ and determine whether each group is CII or CCII.

Order	Group	CII group?	CCII group?	Is nilpotent of class at most 2?
15	$\mathbb{Z}_{15}$	Yes	Yes	Yes
16	$\Lambda_{16}[5]$	Yes	Yes	Yes
16	D_{16}	No	Yes	No
16	Q_{16}	No	Yes	No
16	SD_{16}	No	Yes	No
16	$\mathbb{Z}_2 \times D_8$	Yes	Yes	Yes
16	$\mathbb{Z}_2 \times Q_8$	Yes	Yes	Yes
16	M_{16}	Yes	Yes	Yes
16	$\Gamma_{16,3}$	Yes	Yes	Yes
16	$\Gamma_{16,4}$	Yes	Yes	Yes
16	$\Gamma_{16,13}$	Yes	Yes	Yes

Table 3: The groups of order n with $15 \leq n \leq 16$ (up to isomorphism).

Non-degenerate gyrogroups of order 16

As in Table 3, the dihedral group D_{16}, the generalized quaternion group Q_{16}, and the semidihedral group SD_{16} are CCII, but not nilpotent of class at most 2. Hence, they induce gyrogroups of order 16 having non-trivial gyroautomorphisms. These gyrogroups are pairwise non-isomorphic because D_{16}, Q_{16}, and SD_{16} are pairwise non-isomorphic.

Groups of order n with $n \leq 63$

In [7], all groups of order n with $n \leq 63$ are listed following the classification in [8], and the author determined whether each group is CII or CCII. As a consequence, non-degenerate gyrogroups of orders 32 and 48 arise. The case $n \geq 64$, however, remains open.

[7] T. Suksumran, A note on CII groups and CCII groups, *J. Algebra Comb. Discrete Appl.*, **12**, No. 3 (2025), 259–274.

[8] J. Humphreys, *A course in group theory*, Oxford University Press, Oxford, 1996.

Acknowledgments

Thank you for your attention!

Are there any questions?