The 18th International Conference of Young Algebraists in Thailand
Silpakorn University, Thailand
May 14-16, 2025

Graphical representation of groups and gyrogroups

Teerapong Suksumran

Department of Mathematics

Faculty of Science

Chiang Mai University

Outline of the talk

1. Introduction
2. Cayley graphs
3. Conjugate graphs
4. Gyrogroups
5. Schreier graphs
6. Stanadard Schreier graphs
7. Announcement of the forthcoming conferences
8. Q&A

Introduction

The main theme of the talk is to visualize (abstract) structures, especially groups and gyrogroups, via their graphs and digraphs.

Usually, one can draw a graph and digraph from an abstract group by letting group elements be vertices and by defining edges using some relations involving the group operation.

- Cayley graphs (1878)
- Commuting graphs (1955)
- Intersection graphs (1969)
- Prime graphs (1970)
- Non-commuting graphs (1975)
- Conjugacy class graphs (1990)
- Power graphs (2000)
- Conjugate graphs (2012)

[1] Y. F. Zakariya (2016), Graphs from finite groups: An overview
[2] A. Erfanian and B. Tolue (2012), Conjugate graphs of finite groups

Cayley digraphs and graphs

A Cayley graph associated with a finitely generated group, endowed with the word metric, is the main ingredient in geometric group theory.

Let G be a group with a subset S. The (right) **Cayley digraph** of G with respect to S is the digraph with vertex set G whose arcs are defined by letting that (u, v) is an arc if and only if $v = us$ for some s in S.

The (right) **Cayley graph** of G with respect to S is defined as the underlying graph of the corresponding Cayley digraph, that is, it is the graph with vertex set G whose edges are defined by letting that $\{u, v\}$ is an edge if and only if (u, v) or (v, u) is an arc in the Cayley digraph.

Conjugate graphs

In 2012, Erfanian and Tolue introduced the notion of a conjugate graph associated with a finite non-abelian group [2].

Let G be a finite non-abelian group. The **conjugate graph** of G is the graph with vertex set $G \setminus Z(G)$ whose edges are defined by letting that $\{u, v\}$ is an edge if and only if $v = gug^{-1}$ for some g in G whenever $u \neq v$.

Here, $Z(G)$ denotes the **center** of G defined as

$$Z(G) = \{z \in G : zg = gz \text{ for all } g \in G\}.$$

[2] A. Erfanian and B. Tolue (2012), Conjugate graphs of finite groups

Extended conjugate graphs

In 2022, we extended the notion of a conjugate graph associated with a finite non-abelian group to the notion of an extended conjugate graph associated with a finite group [3].

Let G be a finite group. The **extended conjugate graph** of G is the graph with vertex set G whose edges are defined by letting that $\{u, v\}$ is an edge if and only if $v = gug^{-1}$ for some g in G whenever $u \neq v$.

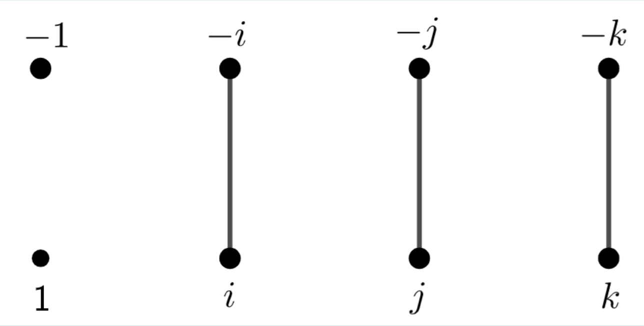


Figure 1: The extended conjugate graph of the quaternion group Q_8.

[3] P. Dangpat and T. S. (2022), Regularity of extended conjugate graphs of finite groups

Cayley digraphs and graphs

A Cayley graph of a given group G encodes a large amount of information about algebraic, combinatorial, and geometric structures of G.

Problem 1. Given a subset A of a finite group G, find the smallest subgroup of G containing A, denoted by $\langle A\rangle$.

Algebraic Tool. We may solve the problem by using an algebraic approach:

$$\langle A\rangle = \left\{a_1^{\varepsilon_1}a_2^{\varepsilon_2}\cdots a_n^{\varepsilon_n} : n\in\mathbb{N}, a_i\in A, \varepsilon_i\in\{-1,1\}\right\}.$$

Cayley digraphs and graphs

Graph-theoretic Tool. We draw the Cayley graph Cay(G, A) and look at the component containing the identity of G.

Example 2. Find $\langle A\rangle$ in the alternating group A_4 of degree 4 when $A = \{(1\ 2)(3\ 4), (1\ 3)(2\ 4)\}$.

The answer is $\langle A\rangle = \{(1), (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$.

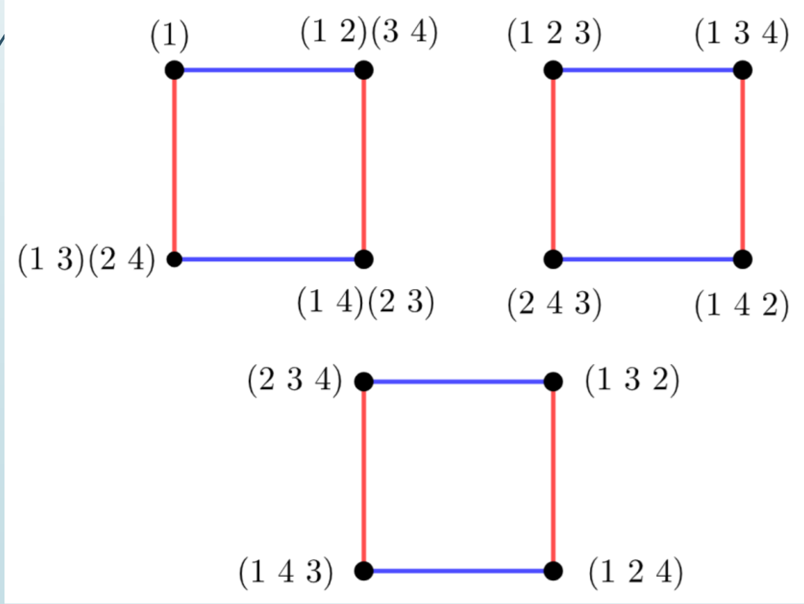


Figure 2: The Cayley graph of A_4 with respect to $A = \{(1\ 2)(3\ 4), (1\ 3)(2\ 4)\}$.

Cayley digraphs and graphs

Theorem 3. Let G be a finite group with $A \subseteq G$. Then the subgroup $\langle A \rangle$ of G equals the set of vertices in the component of $\mathrm{Cay}\,(G, A)$ containing the identity of G. In general, C is a component of $\mathrm{Cay}\,(G, A)$ containing a vertex v if and only if the vertex set of C equals the coset $v\langle A \rangle$.

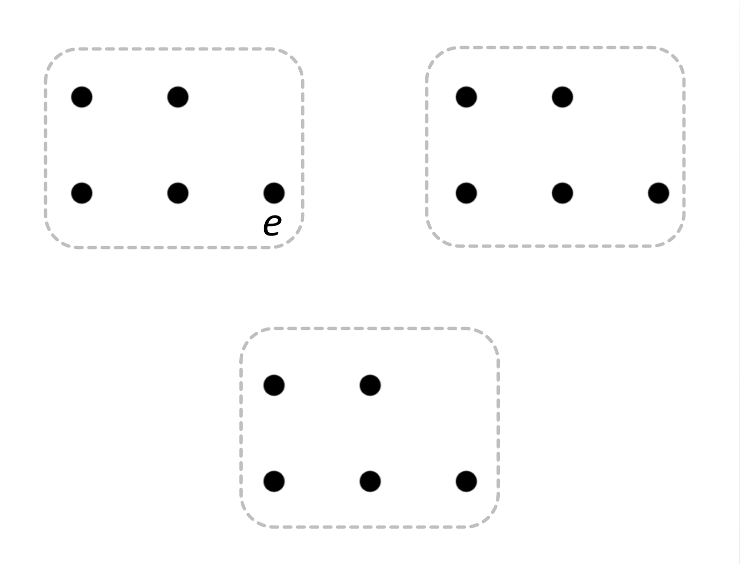


Figure 3: A generic Cayley graph of a finite group.

[4] T. Udomworarat and T. S. (2021), An algorithm for finding minimal generating sets of finite groups

Lagrange's Theorem

Lagrange's Theorem. If H is a subgroup of a finite group G, then the order of H divides the order of G.

In fact, the following **index formula** holds:

$$|G| = |H|[G : H],$$

where $[G : H]$ is the index of H in G, which is the number of distinct left cosets gH with g in G.

A graph-theoretic version of Lagrange’s Theorem

Theorem 4. Let G be a finite group with a subgroup H of G.

(1) Each component of Cay (G, H) has a left coset of H as its vertex set and is the complete graph $K_{|H|}$. There is a one-to-one correspondence between the vertex sets of components of Cay (G, H) and the left cosets of H in G.

(2) Cay (G, H) has $[G : H]$ components. Hence, if H is proper in G, then Cay (G, H) is disconnected.

[4] T. Udomworarat and T. S. (2021), An algorithm for finding minimal generating sets of finite groups

A graph-theoretic version of Lagrange's Theorem

For example, if $H = \{(1), (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$, then there are three cosets of H in the alternating group A_4:

$H = \{(1), (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$,

$(1\ 2\ 3)H = \{(1\ 2\ 3), (1\ 3\ 4), (1\ 4\ 2), (2\ 4\ 3)\}$,

$(1\ 3\ 2)H = \{(1\ 3\ 2), (1\ 2\ 4), (1\ 4\ 3), (2\ 3\ 4)\}$.

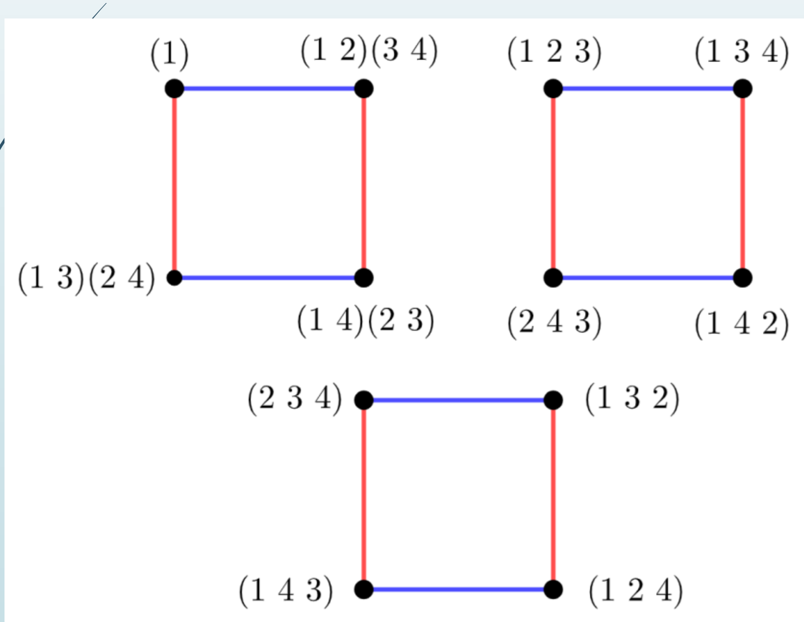


Figure 2: The Cayley graph of A_4 with respect to $A = \{(1\ 2)(3\ 4), (1\ 3)(2\ 4)\}$.

Normal subgroups VS bipartite graphs

Theorem 5. Let G be a finite group with a symmetric generating set S for G not containing the identity of G. Then $\mathrm{Cay}(G, S)$ is bipartite if and only if G has a subgroup of index 2 that is disjoint from S.

For example, we know that D_8 has a normal subgroup of index 2 disjoint from $\{r, r^3, s\}$, which is $\{1, r^2, rs, r^3s\}$, by drawing the following Cayley graph.

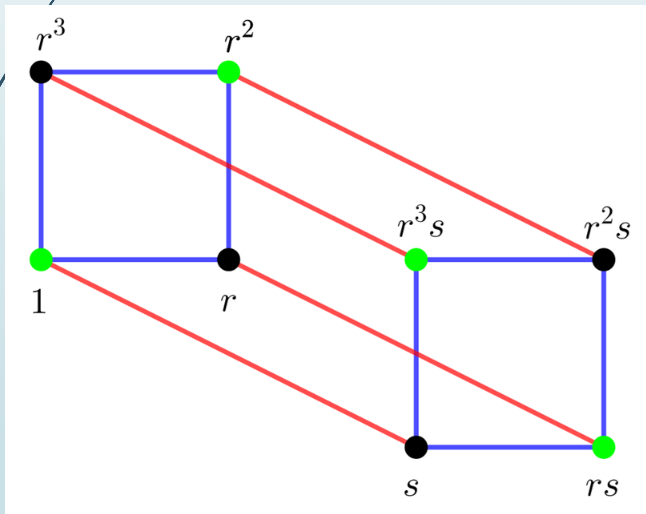


Figure 4: The Cayley graph $\mathrm{Cay}(D_8, \{r, r^3, s\})$, where D_8 is the dihedral group of order 8.

[5] R. van Dam and M. Jazaeri (2022), On bipartite distance-regular Cayley graphs with small diameter

Normal subgroups VS bipartite graphs

Moreover, we know that any Cayley graph Cay(A_4, S), where S is a symmetric generating set for A_4 not containing the identity, is never bipartite since, as a standard result in abstract algebra, A_4 has no subgroup of order 6.

In summary, Theorem 5 is a nice example of a result that enables us to understand a group structure by considering its corresponding Cayley graph.

Extended conjugate graphs

An extended conjugate graph of a given group G reflects information about the algebraic structure of G in a surprising way.

Let G be a finite group. The **extended conjugate graph** of G is the graph with vertex set G whose edges are defined by letting that $\{u, v\}$ is an edge if and only if $v = gug^{-1}$ for some g in G whenever $u \neq v$.

If G is a finite non-abelian group, then the extended conjugate graph of G is the union of the conjugate graph of G and the isolated vertices corresponding to the elements of $Z(G)$.

[3] P. Dangpat and T. S. (2022), Regularity of extended conjugate graphs of finite groups

Probability of element commutativity

Let G be a finite group. Denote by $R(G)$ the **probability** that a pair of elements of G, chosen at random in G, will commute with each other. It is known in the literature that $R(G) \leq 5/8$ if G is non-abelian [6].

Theorem 6. Let G be a finite group. Then

$$R(G) = \frac{\text{the number of components of the extended conjugate graph of } G}{\text{the number of vertices of the extended conjugate graph of } G}.$$

[6] D. MacHale (1974), How commutative can a non-commutative group be?
[3] P. Dangpat and T. S. (2022), Regularity of extended conjugate graphs of finite groups

Probability of element commutativity

For example, by drawing the extended conjugate graph of the quaternion group Q_8, we conclude that $R(Q_8) = 5/8$.

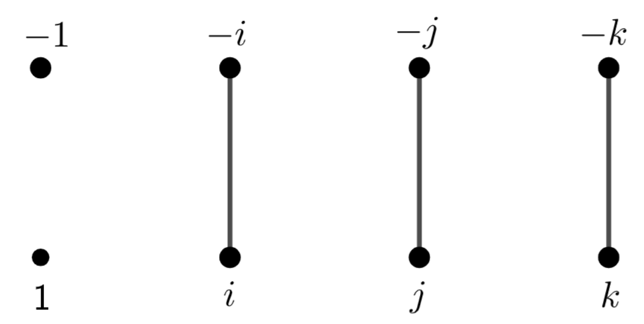


Figure 1: The extended conjugate graph of the quaternion group Q_8.

This also shows that the upper bound 5/8 is sharp.

The class equation

Let G be a finite group. Recall that the **class equation** for G is

$$|G| = |Z(G)| + \sum_{1\leq i\leq n}[G : C_G(g_i)],$$

where $C_G(g_i)$ is the centralizer of g_i in G defined as

$$C_G(g_i) = \{g\in G : gg_ig^{-1} = g_i\}$$

and g_i is not in $Z(G)$ for all i (if any).

A graph-theoretic version of the class equation

Proposition 7. Let G be a finite group, and let $g \in G$. If Ξ is a component of the extended conjugate graph of G containing vertex g, then $V(\Xi)$ equals the conjugacy class of g, $|V(\Xi)| = [G : C_G(g)]$, and Ξ is a complete graph.

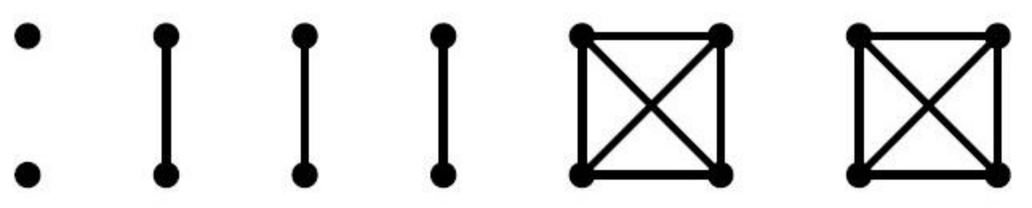


Figure 5: The extended conjugate graph of the dihedral group D_{16}.

[3] P. Dangpat and T. S. (2022), Regularity of extended conjugate graphs of finite groups

Nilpotency VS regularity

Recall that a group G is **nilpotent** if its upper central series reaches G at some point. As an example, every finite p-group is nilpotent.

Recall also that a graph is **regular** if every vertex has the same degree. As an example, the Cayley graph of A_4 with respect to $A = \{(1\ 2)(3\ 4), (1\ 3)(2\ 4)\}$ is regular.

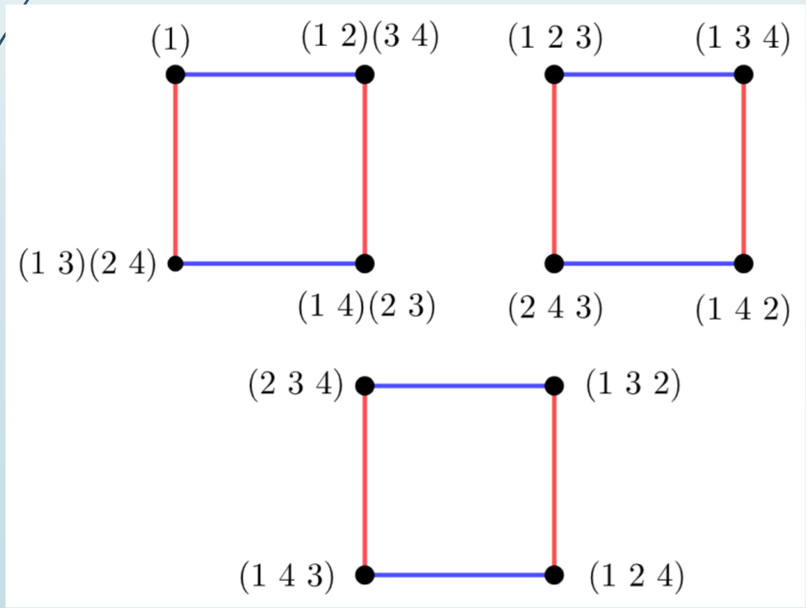


Figure 2: The Cayley graph of A_4 with respect to $A = \{(1\ 2)(3\ 4), (1\ 3)(2\ 4)\}$.

Nilpotency VS regularity

Theorem 8. Let G be a finite non-abelian group. If the conjugate graph of G is regular, then G is nilpotent.

The converse to Theorem 8 is not, in general, true. In fact, the dihedral group D_{16} is nilpotent since it is a 2-group. However, the conjugate graph of D_{16} is not regular.

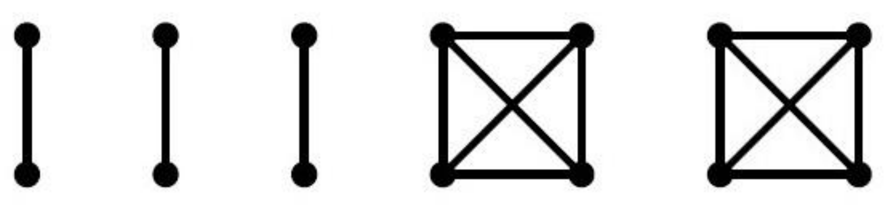


Figure 6: The conjugate graph of the dihedral group D_{16}.

[3] P. Dangpat and T. S. (2022), Regularity of extended conjugate graphs of finite groups

22

Centerlessness VS non-regularity

Theorem 9. Let G be a finite non-trivial group. If G is centerless (i.e., $Z(G) = \{e\}$), then the conjugate graph of G is not regular.

Consequently, the conjugate graph of the following group is not regular:

- the symmetric group S_n for all $n \geq 3$;
- the dihedral group D_{2n} for all odd integers $n \geq 3$;
- any non-abelian finite simple group;
- any Frobenius group

because its center is trivial.

[3] P. Dangpat and T. S. (2022), Regularity of extended conjugate graphs of finite groups

Combination of Cayley and extended conjugate graphs

Cayley graphs and extended conjugate graphs can be thought of as graphs associated with group actions.

Recall that a group G acts on itself by left multiplication: $g{\cdot}x = gx$ for all $g, x \in G$. This action induces a (left) Cayley graph of G with respect to G.

Recall also that a group G acts on itself by conjugation: $g{\cdot}x = gxg^{-1}$ for all $g, x \in G$. This action induces the extended conjugate graph of G.

From this point of view, one can construct a graph from a group action on a non-empty set. We will do so in a more general setting.

A prime example of a gyrogroup

The theory of gyrogroups is originated from the study of parametrization of Lorentz transformations in physics and has a strong connection with Einstein's velocity addition.

A prime example of a gyrogroup is the (complex) **Möbius gyrogroup**, which consists of the open unit disk in the complex plane,

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\},$$

together with **Möbius addition** defined by

$$a \oplus_M b = \frac{a+b}{1+\bar{a}b}$$

for all $a, b \in \mathbb{D}$.

The formal definition of a gyrogroup

Modeled on Einstein's velocity addition and Möbius addition, Ungar formulated the formal definition of a gyrogroup.

We slightly modify Ungar's definition as follows:

Let G be a non-empty set endowed with a binary operation $\oplus$. The pair $(G, \oplus)$ is called a **gyrogroup** if

(1) there is an element $e \in G$ for which $e \oplus a = a$ for all $a \in G$;
(2) for each $a \in G$, there is an element $b \in G$ for which $b \oplus a = e$;
(3) for all $a, b \in G$, there is an automorphism of $(G, \oplus)$, written $\mathrm{gyr}[a, b]$, such that

$$a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a, b](c)$$

for all $c \in G$;
(4) for all $a, b \in G$, $\mathrm{gyr}[a \oplus b, b] = \mathrm{gyr}[a, b]$.

Gyrogroup actions and permutation representations

The notion of group actions can be extended to the setting of gyrogroups in a natural way.

Let G be a gyrogroup, and let X be a non-empty set. A function $\cdot$ from $G \times X$ to X, where $\cdot(g, x)$ is written by $g \cdot x$, is called a **gyrogroup action** of G on X if

(1) $e \cdot a = a$ for all $a \in G$; and

(2) $a \cdot (b \cdot x) = (a \oplus b) \cdot x$ for all $a, b \in G$ and for all $x \in X$.

Every gyrogroup action of G on X induces a homomorphism from G to $\mathrm{Sym}(X)$, where $\mathrm{Sym}(X)$ is viewed as a gyrogroup with trivial gyroautomorphisms, and vice versa. This homomorphism is referred to as a **permutation representation** of G on X.

[7] T. S. (2016), Gyrogroup actions: A generalization of group actions

Example of a gyrogroup action

In the gyrogroup of order 16, K_{16}, its coset space induced by the subgyrogroup $H = \{0, 1\}$ has eight disjoint cosets:
$X_1 = \{0, 1\}$, $X_2 = \{2, 3\}$, $X_3 = \{4, 5\}$, $X_4 = \{6, 7\}$, $X_5 = \{8, 9\}$, $X_6 = \{10, 11\}$, $X_7 = \{12, 13\}$, $X_8 = \{14, 15\}$.

As H satisfies a certain condition, K_{16} acts on $K_{16}/H = \{X_1, X_2, \ldots, X_8\}$ by left gyroaddition:

$$g\cdot(x \oplus H) = (g \oplus x) \oplus H$$

for all $g, x \in K_{16}$.

[8] T. S. (2022), On Schreier graphs of gyrogroup actions

Schreier digraphs and graphs

We can construct a digraph and graph from a gyrogroup action, generalizing Cayley graphs and extended conjugate graphs.

Suppose that a gyrogroup G acts on a finite non-empty set X. Let $A \subseteq G$. The **Schreier digraph** of G with respect to A is defined as the digraph with vertex set X and arc set $\{(x, a \cdot x) : x \in X, a \in A\}$.

Suppose that a gyrogroup G acts on a finite non-empty set X. Let $A \subseteq G$. The **Schreier graph** of G with respect to A is defined as the underlying graph of the Schreier digraph of G with respect to A: the vertex set is X, and $\{u, v\}$ is an edge in the Schreier graph if and only if (u, v) or (v, u) is an arc in the Schreier digraph.

[8] T. S. (2022), On Schreier graphs of gyrogroup actions

Schreier digraphs and graphs

As K_{16} acts on $K_{16}/H = \{X_1, X_2, \ldots, X_8\}$ by left gyroaddition, its corresponding Schreier digraph and graph with respect to $A = \{4, 8\}$ are depicted below.

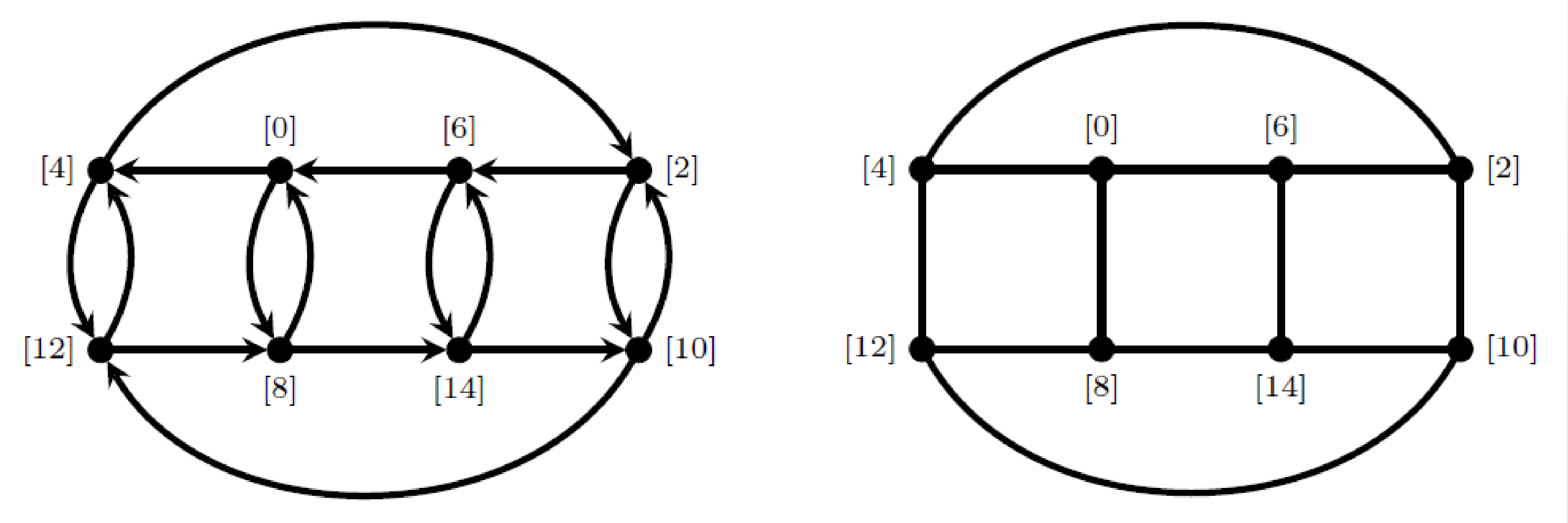


Figure 7: The Schreier digraph and graph of K_{16} with respect to $A = \{4, 8\}$.

[8] T. S. (2022), On Schreier graphs of gyrogroup actions

Transitivity VS connectedness

Algebraic properties of gyrogroup actions are reflected in graph-theoretic properties of Schreier digraphs and graphs.

Theorem 10. Let G be a gyrogroup, let $\emptyset \neq A \subseteq G$, and let X be a finite G-set. If the Schreier graph with respect to A is connected, then the action of G on X is transitive. The converse holds if $A \cup \ominus A$ is a left generating set for G.

For example, the action of K_{16} on K_{16}/H is transitive since its Schreier graph is connected (see Figure 7).

[8] T. S. (2022), On Schreier graphs of gyrogroup actions

The Orbit-Stabilizer Theorem

The Orbit-Stabilizer Theorem for gyrogroup actions, which generalizes the well-known Orbit-Stabilizer Theorem in abstract algebra is stated below.

The Orbit-Stabilizer Theorem. If a gyrogroup G acts on a non-empty set X, then

$$|\mathrm{orb}(x)| = [G : \mathrm{stab}(x)]$$

for all $x \in X$. In the case when G is finite,

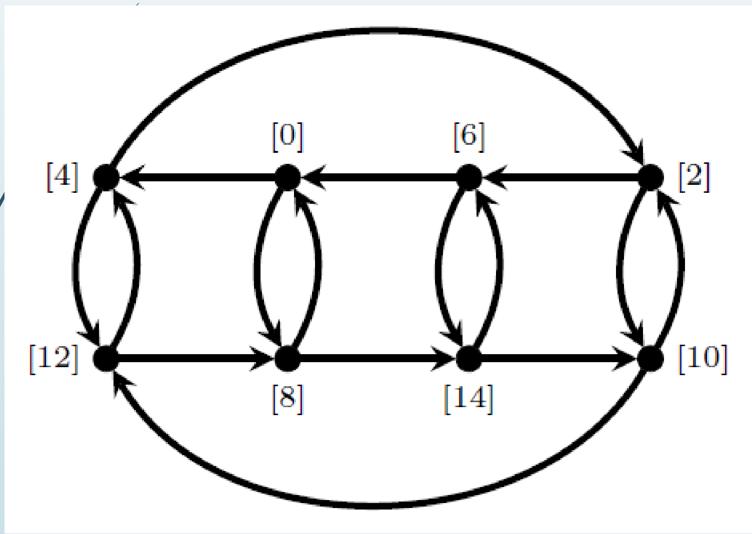
$$|G| = |\mathrm{orb}(x)||\mathrm{stab}(x)|.$$

Here, $\mathrm{orb}(x) = \{a \cdot x : a \in G\}$ and $\mathrm{stab}(x) = \{g \in G : g \cdot x = x\}$.

[7] T. S. (2016), Gyrogroup actions: A generalization of group actions

A graph-theoretic version of the Orbit-Stabilizer Theorem

Theorem 11. Suppose that a gyrogroup G acts on a finite non-empty set X, and let A be a finite non-empty subset of G. For each $x \in X$, the out-degree of x in the Schreier digraph with respect to A is $[A : \mathrm{stab}(x)]$ and the in-degree of x in the Schreier digraph with respect to A is $[\ominus A : \mathrm{stab}(x)]$.



For example, from Figure 8, we know that $[A : \mathrm{stab}(x)] = 2$ for all x in $K_{16}/\{0, 1\}$.

Figure 8: The Schreier digraph of K_{16} with respect to $A = \{4, 8\}$.

[8] T. S. (2022), On Schreier graphs of gyrogroup actions

Gyrocommutativity VS balancedness

The notion of gyrocommutativity generalizes that of commutativity. Recall that a gyrogroup G is said to be **gyrocommutative** if

$$a \oplus b = \operatorname{gyr}[a, b](b \oplus a)$$

for all $a, b \in G$ [9]. Hence, every abelian group is gyrocommutative.

Recall also that a digraph is said to be **balanced** if the in-degree and out-degree of each vertex equal.

Theorem 12. Suppose that a gyrogroup G acts on a non-empty set X. If G is gyrocommutative, then the Schreier digraph with respect to A is a balanced digraph for all subsets A of G.

[8] T. S. (2022), On Schreier graphs of gyrogroup actions
[9] A. Ungar (2008), Analytic Hyperbolic Geometry and Albert Einstein's Special Theory of Relativity

Gyrogroup actions and group actions

There is a strong connection between a gyrogroup action and group action.

Suppose that a gyrogroup G acts on a non-empty set X. Then the kernel K of the action, $K = \{g \in G : g\cdot x = x \text{ for all } x \in X\}$ forms a normal subgyrogroup of G so that G/K forms a quotient gyrogroup isomorphic to a subgroup of $\mathrm{Sym}(X)$. Hence, G/K is a group that acts on X by the same formula

$$(g \oplus K)\cdot x = g\cdot x$$

for all $g \in G$, $x \in X$. In this case, gyrogroup orbits and group orbits coincide, and gyrogroup stabilizers are quotient group stabilizers.

Gyrogroup actions and group actions

This fact leads to the notion of standard Schreier digraphs and graphs.

Let $A \subseteq S_n$. The **standard Schreier digraph** with respect to A is the digraph with vertex set $I = \{i \in \mathbb{N} : i \leq n\}$ and arc set $\{(i, \sigma(i)) : i \in \mathrm{I}, \sigma \in A\}$.

The **standard Schreier graph** with respect to A is defined as the underlying graph of the standard Schreier digraph with respect to A.

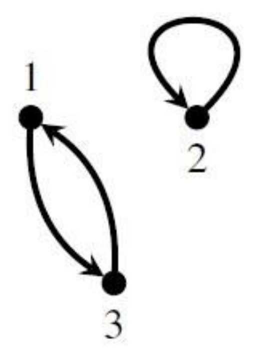


Figure 9: The Schreier digraph of S_3 with respect to $A = \{(1\ 3)\}$.

Standard Schreier digraphs and Schreier digraphs

The importance of standard Schreier digraphs lie in the following theorem.

Theorem 13. Suppose that a gyrogroup G acts on a finite non-empty set X, and let $A \subseteq G$. Then the Schreier digraph with respect to A is isomorphic to a standard Schreier digraph of S_n for some $n \in \mathbb{N}$.

[8] T. S. (2022), On Schreier graphs of gyrogroup actions

Questions & Answers

Q&A