


# Computational Tools for Measuring Non-Associativity in Einstein Velocity Addition

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## ABSTRACT

Using associators, we develop measures of the non-associativity of Einstein addition on the open unit ball in Euclidean space $\mathbb{R}^n$. Our results show that these measures characterize departures from the associative law and reveal algebraic and geometric features of Einstein addition.

## OBJECTIVE

- To develop associator-based measures for Einstein addition on the open unit ball in $\mathbb{R}^n$.
- To use a compact formula to quantify the non-associativity of Einstein addition.

## METHODOLOGY

Einstein addition is generally non-associative: regrouping three velocities can change their sum. We use associators to measure this discrepancy.

**1 Formulate associators**

Analyze Einstein addition using algebraic techniques.

**2 Define quantitative indices**

Measure departures from the associative law.

**3 Run large-scale simulations**

Use software to map associator values on the unit ball.

## BASIC DEFINITION

Let $\mathbb{B} = \{\mathbf{v} \in \mathbb{R}^n : \|\mathbf{v}\| < 1\}$ be the open unit ball in $\mathbb{R}^n$. With the speed of light normalized to 1, Einstein addition $\oplus_E$ on $\mathbb{B}$ is given by the formula

$$\mathbf{u} \oplus_E \mathbf{v} = \frac{1}{1+\mathbf{u}\cdot\mathbf{v}}\left(\mathbf{u} + \frac{1}{\gamma_\mathbf{u}}\mathbf{v} + \frac{\gamma_\mathbf{u}}{1+\gamma_\mathbf{u}}(\mathbf{u}\cdot\mathbf{v})\,\mathbf{u}\right),$$

where the Lorentz factor is given by

$$\gamma_\mathbf{u} = \frac{1}{\sqrt{1-\|\mathbf{u}\|^2}}.$$

Here, $\cdot$ denotes the Euclidean inner product on $\mathbb{R}^n$, and $\mathbf{u}, \mathbf{v} \in \mathbb{B}$.

The associator of $(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathbb{B}^3$ is defined as

$$[\mathbf{u}, \mathbf{v}, \mathbf{w}] = -(\mathbf{u} \oplus_E (\mathbf{v} \oplus_E \mathbf{w})) \oplus_E ((\mathbf{u} \oplus_E \mathbf{v}) \oplus_E \mathbf{w}).$$

## MAIN RESULT

### Theorem (Uniqueness and Vanishing Criteria for Associators)

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be arbitrary vectors in $\mathbb{B}$.

1. The associator $[\mathbf{u}, \mathbf{v}, \mathbf{w}]$ is the unique vector in $\mathbb{B}$ for which
$$(\mathbf{u} \oplus_E \mathbf{v}) \oplus_E \mathbf{w} = (\mathbf{u} \oplus_E (\mathbf{v} \oplus_E \mathbf{w})) \oplus_E [\mathbf{u}, \mathbf{v}, \mathbf{w}]. \qquad (\star)$$
2. If one of $\mathbf{u}$, $\mathbf{v}$, or $\mathbf{w}$ is the zero vector, then $[\mathbf{u}, \mathbf{v}, \mathbf{w}] = \mathbf{0}$.
3. $[\mathbf{u}, \mathbf{v}, \mathbf{w}] = \mathbf{0}$ if and only if $(\mathbf{u} \oplus_E \mathbf{v}) \oplus_E \mathbf{w} = \mathbf{u} \oplus_E (\mathbf{v} \oplus_E \mathbf{w})$.
4. $[\mathbf{u}, \mathbf{v}, \mathbf{w}] = \mathbf{0}$ if and only if $\mathbf{u}\cdot\mathbf{w} = \mathbf{v}\cdot\mathbf{w} = 0$ or $\mathbf{u} \parallel \mathbf{v}$.

The associator $[\mathbf{u}, \mathbf{v}, \mathbf{w}]$ tests associativity of each triple: it vanishes exactly when $\mathbf{w}$ is orthogonal to $\operatorname{span}\{\mathbf{u}, \mathbf{v}\}$ or $\mathbf{u}$ and $\mathbf{v}$ are collinear.

## RESULT VISUAL

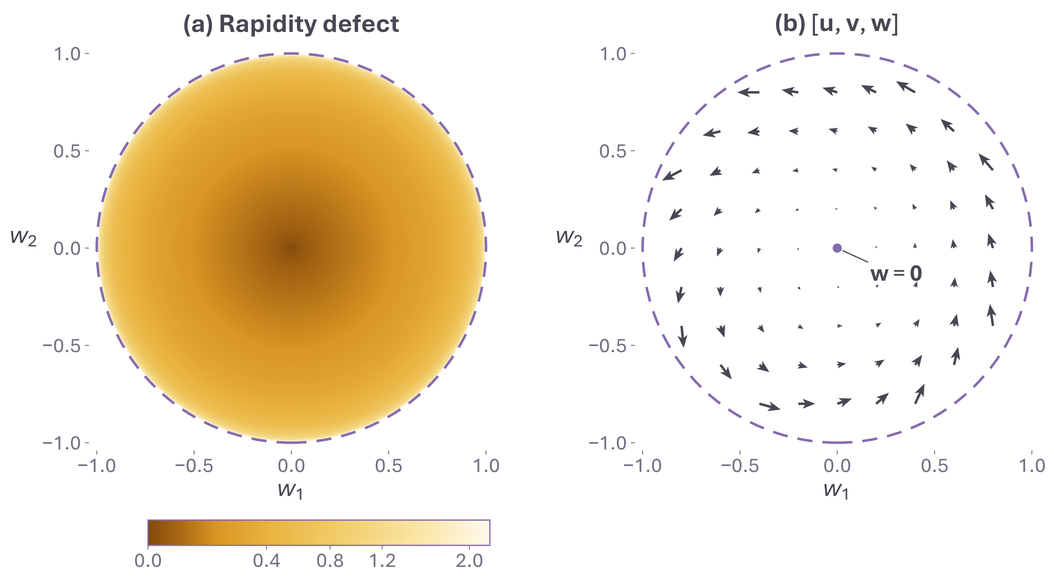



Figure 1. Fix $\mathbf{u} = (0.65, 0)$ and $\mathbf{v} = (0, 0.65)$. (a) Heatmap representing the rapidity defect $\tanh^{-1}\|[\mathbf{u}, \mathbf{v}, \mathbf{w}]\|$; (b) vector field depicting $[\mathbf{u}, \mathbf{v}, \mathbf{w}]$. On the two-dimensional disk, the rapidity defect has concentric circular contours and vanishes inside the disk if and only if $\mathbf{w} = \mathbf{0}$.

## CONCLUSION & RELEVANCE

Every triple $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ of relativistically admissible velocities has a unique associator measuring non-associativity. It vanishes exactly when $\mathbf{u}$ and $\mathbf{v}$ are collinear or $\mathbf{w}$ is orthogonal to both. The rapidity defect makes this criterion visible and provides a computational tool for studying relativistic velocity addition.

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## DECLARATION OF AI USE

AI tools assisted with drafting, language editing, figure code, and formula checks; the author is responsible for the content.

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